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Perturbation of a dynamic planar crack moving in a model viscoelastic solid

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Abstract

Weight functions, which give stress intensity factors in terms of applied loading, are constructed, for three-dimensional time-dependent loading of a semi-infinite crack, propagating at uniform speed. Both a model problem, governed by a scalar wave equation, and the full vectorial problem for Mode I loading, are considered. The medium through which the crack propagates is viscoelastic; the approach is general but explicit formulae are given when the medium is a Maxwell fluid. The weight functions are exploited to develop formulae for the first-order perturbations of stress intensity factors when the crack edge is no longer straight but becomes slightly wavy. Implications for stability, and for "crack front waves" in the case of the Mode I problem, are discussed.

Keywords: Crack propagation; Dynamic weight functions; Viscoelastic medium; Wavy crack; Crack front stability

1. Introduction

Several studies have been made concerning the perturbation from straight of a crack, induced by regions of locally heterogeneous resistance to fracture. Rice et al. (1994) addressed the problem of such a crack moving through a model elastic solid, in which motion was governed by a single scalar wave equation in place of the full vectorial equations of elastodynamics. Perrin and Rice (1994) used these results to show that no statistically stationary crack configurations exist under small random variations in critical fracture energy. Morrissey and Rice (1998) presented simulations of 3D dynamic fracture, revealing the existence of a decaying signal propagating along the crack front in the model problem, and a persistent "crack front wave" in the case of Mode I loading in real vectorial elastodynamics. These phenomena arise from a singularity on the real axis in the Fourier transform of the transfer function relating crack front position to fracture energy. This singularity exists both in the model problem (Rice et al., 1994), and in the Mode I problem (Ramanathan and Fisher, 1997).

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The above studies all relate to a case in which the coefficient of the *second* term in the asymptotic expansion of the stress ahead of the unperturbed crack (denoted m) is zero. Woolfries and Willis (1999) addressed the scalar model problem to take account of possible non-zero m. They showed that the results of Willis and Movchan (1995) agreed with those of previously mentioned studies when m was zero. They went on to show that perturbations in crack front position grew exponentially with time when m was positive, and decayed exponentially when m was negative. This corresponds to the sign of the imaginary part of the position of the singularity in the transfer function in complex transform space. The case of zero m, giving rise to a real singularity, is thus at the boundary of stability: "crack front disorder" grows algebraically rather than exponentially with time.

Very few exact closed form solutions exist relating to viscoelastic fracture problems. Willis (1967) used a Wiener–Hopf factorization to obtain the stress intensity factor for a steady-state semi-infinite anti-plane shear crack propagating through a standard linear solid. Walton (1982) generalized these results to a general viscoelastic medium. Atkinson and Coleman (1977), Atkinson (1979) and Atkinson and Popelar (1979) have also addressed the dynamic fracture of linear viscoelastic material.

This paper considers the effect of dissipation on a perturbed crack moving through one of the simplest viscoelastic materials, a Maxwell fluid. The Fourier transform of the transfer function relating crack front position to fracture energy is found for both the scalar and vectorial Mode I problems. In the scalar model case (Section 2), this transform and its complex singularities are found explicitly. Crack front stability depends on a balance between the coefficient m and the relaxation time τ. Increasing the amount of dissipation through decreasing the relaxation time τ increases the crack stability. The stability criterion relating these variables is found explicitly. The limiting behaviour of the transfer function in real space near the wavefront is also found explicitly. The transfer function is found from its transform numerically and plotted using the discrete Fourier transform.

The Mode I problem is also considered (Section 3), following closely the work of Willis and Movchan (1995). An expression for the Fourier transform of the transfer function is given; it contains an integral which must be evaluated numerically. It is found that the singularity present in elasticity which gives rise to crack front waves (found by Ramanathan and Fisher, 1997) is moved off the real axis by the introduction of viscoelastic dissipation, rendering the crack stable for zero m. Repeating the process for a range of values shows that, as for the scalar case, crack stability increases with the amount of dissipation.

2. Scalar model problem

We begin with analysis of a scalar problem for a crack propagating through a Maxwell fluid.

2.1. Problem formulation

The problem to be solved is the model problem introduced by Rice et al. (1994), now governed by a viscoelastic constitutive relation characterized by a relaxation function whose time derivative is a generalized function, denoted $L(t)$. The model solid is described by a single "displacement" variable $u(t, x_1, x_2, x_3)$, and an associated "stress" variable $\sigma(t, x_1, x_2, x_3)$ given by

$$\sigma(t, \mathbf{x}) = \int L(t - t')e(t', \mathbf{x})\,\mathrm{d}t', \tag{2.1}$$

where

$$e(t, \mathbf{x}) = \frac{\partial u}{\partial x_3}(t, \mathbf{x}). \tag{2.2}$$

The equation of motion for the displacement is

$$\int L(t-t')\nabla^2 u(t',\mathbf{x})\,\mathrm{d}t' = \rho u_{,tt}(t,\mathbf{x}), \tag{2.3}$$

where ρ is the density.

A crack propagates dynamically so that, at time t, it occupies the domain

$$S_\epsilon = \{-\infty < x_1 - Vt < \epsilon\phi(t,x_2), -\infty < x_2 < \infty, x_3 = 0\}, \tag{2.4}$$

where $0 \leqslant \epsilon \ll 1$. The crack front at time t therefore lies along the curve

$$x_1 = a(t,x_2) := Vt + \epsilon\phi(t,x_2), \quad x_3 = 0. \tag{2.5}$$

The boundary condition

$$\sigma = 0 \quad \text{on } x_3 = 0, \quad x_1 < a(t,x_2) \tag{2.6}$$

is to be enforced, with general remote loading conditions. In the text below we derive the asymptotic formulae for the stress intensity factor and the energy release rate associated with the moving crack.

2.2. Fundamental identity and the weight function

Fourier transforming (2.3) with respect to the variables x_1, x_2 and t, the function $\tilde{u}(\omega,\xi_1,\xi_2,x_3)$, defined by

$$\tilde{u}(\omega,\xi_1,\xi_2,x_3) = \int\int\int u(t,x_1,x_2,x_3)\exp(\mathrm{i}\omega t)\exp(\mathrm{i}\xi_1 x_1)\exp(\mathrm{i}\xi_2 x_2)\,\mathrm{d}t\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{2.7}$$

satisfies the equation

$$\frac{\partial^2\tilde{u}}{\partial x_3^2} = \left(\xi_1^2 + \xi_2^2 - \frac{\omega^2}{(c(\omega))^2}\right)\tilde{u}, \tag{2.8}$$

where

$$(c(\omega))^2 = \frac{\widetilde{L}(\omega)}{\rho}, \tag{2.9}$$

and $\widetilde{L}(\omega)$ is the time Fourier transform of $L(t)$. A solution which decays as $|x_3| \to \infty$ is

$$\tilde{u}(\omega,\xi_1,\xi_2,x_3) = B(\omega,\xi_1,\xi_2)\exp\left(\mathrm{i}\left(\omega^2/(c(\omega))^2 - \xi_1^2 - \xi_2^2\right)^{1/2}|x_3|\right), \tag{2.10}$$

and, looking at (2.1), the transform of the stress variable is

$$\tilde{\sigma}(\omega,\xi_1,\xi_2,x_3) = \mathrm{i}\widetilde{L}(\omega)\mathrm{sgn}(x_3)\left(\omega^2/(c(\omega))^2 - \xi_1^2 - \xi_2^2\right)^{1/2}\tilde{u}(\omega,\xi_1,\xi_2,x_3). \tag{2.11}$$

It is desirable to introduce a moving coordinate,

$$X := x_1 - Vt. \tag{2.12}$$

Then, following Willis and Movchan (1995), we define the field $U(t,x_1,x_2,x_3)$ and its associated traction $\Sigma(t,x_1,x_2,x_3)$ so as to satisfy the equation of motion (2.3) and the conditions

$$\begin{aligned}
&\Sigma(t, Vt+X, x_2, 0) = 0 \quad \text{when } X > 0,\\
&[\Sigma](t, X, x_2) = 0 \quad \text{for all } X,\\
&[U](t, X, x_2) = 0 \quad \text{when } X < 0,\\
&[U](t, X, x_2) \to (2/\pi)^{1/2} X^{-1/2}\delta(x_2)\delta(t) \quad \text{as } X \to 0_+.
\end{aligned} \tag{2.13}$$

Here, the notation $[\cdot]$ means the jump across $x_3 = 0$ of the quantity indicated, evaluated when $x_1 = Vt + X$. The justification for this definition, in the context of viscoelastic response, is given in Appendix A.

From (2.11), the Fourier transforms of the displacement jump $[U]$ across the plane $x_3 = 0$ and the traction Σ (interpreted via (2.12) as functions of t, x_1 and x_2) are related by

$$\widetilde{\Sigma}(\omega', \xi_1, \xi_2) = \frac{\mathrm{i}}{2}\widetilde{L}(\omega')\Big((\omega')^2/(c(\omega'))^2 - \xi_1^2 - \xi_2^2\Big)^{1/2}[\widetilde{U}](\omega', \xi_1, \xi_2). \tag{2.14}$$

Here, we set $\omega' = \omega - V\xi_1$. This allows the Fourier transforms, regarded as functions of (ω, ξ_1, ξ_2), to be interpreted as transforms relative to t, X and x_2 (Willis, 1997, 2000). Explicitly, if $\overline{(\cdot)}$ denotes the transform relative to variables t, X and x_2,

$$\bar{f}(\omega, \xi_1, \xi_2) = \int \mathrm{d}t \int \mathrm{d}X \int \mathrm{d}x_2 f(t, Vt+X, x_2)\exp\{\mathrm{i}[\omega t + \xi_1 X + \xi_2 x_2]\}, \tag{2.15}$$

where $f(t, x_1, x_2)$ is defined relative to the fixed frame. It is easily verified that

$$\bar{f}(\omega, \xi_1, \xi_2) = \tilde{f}(\omega - V\xi_1, \xi_1, \xi_2). \tag{2.16}$$

This elementary observation allows the immediate extension to viscoelasticity of the elastic analysis of Willis and Movchan (1995).

Eq. (2.14) is true for any constitutive function L. From now on, we specialize to a Maxwell fluid with constitutive relation given by

$$\dot{\sigma} + \frac{\sigma}{\tau} = E\dot{e}, \tag{2.17}$$

where the superposed dot denotes $(\partial/\partial t)$, so that $L(t) = E[\delta(t) - \tau^{-1}\exp(-t/\tau)H(t)]$ and has Fourier transform

$$\widetilde{L}(\omega) = \frac{E\omega\tau}{\omega\tau + \mathrm{i}}. \tag{2.18}$$

Thus, $\widetilde{L}(\omega')$ is analytic for ξ_1 in the lower half of the complex ξ_1 plane when ω is real or has positive imaginary part, as is usually invoked for causality. For this constitutive relation, the phase speed $c(\omega)$ is given by

$$c(\omega) = d[1 + \mathrm{i}/\omega\tau]^{-1/2}, \tag{2.19}$$

where $d := (E/\rho)^{1/2}$ is the elastic, high-frequency, wave speed. The "square root" function in (2.14) is easily factorized,

$$\Big((\omega')^2/(c(\omega'))^2 - \xi_1^2 - \xi_2^2\Big)^{1/2} = \mathrm{i}\delta(\xi_1 - \xi_d^+)^{1/2}(\xi_1 - \xi_d^-)^{1/2}, \tag{2.20}$$

where

$$\xi_d^{\pm}(\omega, \xi_2) = -\frac{V\omega}{\delta^2 d^2} - \frac{\mathrm{i}V}{2\delta^2 d^2\tau} \pm \left(\frac{\omega^2}{\delta^4 d^2} + \frac{\mathrm{i}\omega}{\delta^4 d^2\tau} - \frac{V^2}{4\delta^4 d^4\tau^2} - \frac{\xi_2^2}{\delta^2}\right)^{1/2} \tag{2.21}$$

and the constant δ is given by $\delta^2 = 1 - V^2/d^2$. If the square root function in (2.21) is defined so as to have positive imaginary part when ω and ξ_2 are real, then ξ_d^+ has positive imaginary part, and ξ_d^- has negative imaginary part when ω and ξ_2 are real. The proof of this is as follows. We know from the work of Willis and Movchan (1995) that in the limit $\tau \to \infty$ (corresponding to elasticity), the two branch points $\xi_d^\pm$ are in different half-planes, possibly separated by the (vanishingly small) imaginary part of ω. By rearranging (2.20), it can be shown that $\xi_d^\pm$ satisfies

$$\omega - V\xi_d^\pm = -\frac{\mathrm{i}}{2\tau} \pm \left(d^2(\xi_d^{\pm^2} + \xi_2^2) - \frac{1}{4\tau^2}\right)^{1/2}. \tag{2.22}$$

Suppose that $\xi_d^\pm$ is real for some real ω. Then, both sides of (2.22) are real. This can only be true if $\omega = \xi_d^+ = \xi_2 = 0$. It is never possible for ξ_d^- to be real, if ω has non-negative imaginary part. Hence, $\xi_d^\pm$ never cross the real axis as τ varies, so they always remain in their respective half-planes. It should be noted that, for general τ, $\xi_d^\pm$ are away from the real axis even when ω is exactly real, thus removing the need for a small imaginary addition to ω.

The equation (2.14) may be rewritten as

$$\frac{2\overline{\Sigma}}{\delta\widetilde{L}(\omega')(\xi_1 - \xi_d^+)^{1/2}} = -(\xi_1 - \xi_d^-)^{1/2}[\overline{U}]. \tag{2.23}$$

The left side of (2.23) is analytic in a lower half of the complex ξ_1 plane, and the right hand side is analytic in an upper half-plane. Therefore, by analytic continuation, the right hand side describes an entire function, which must be constant, by Liouville's theorem, since by the conditions imposed, $[\overline{U}] \sim (2\mathrm{i})^{1/2}(\xi_1 + 0\mathrm{i})^{-1/2}$ as $\xi_1 \to \infty$. Hence,

$$[\overline{U}](\omega, \xi_1, \xi_2) = (2\mathrm{i})^{1/2}(\xi_1 - \xi_d^-)^{-1/2}. \tag{2.24}$$

Expanding $[\overline{U}]$ for large $|\xi_1|$,

$$[\overline{U}] = \frac{(2\mathrm{i})^{1/2}}{(\xi_1 + 0\mathrm{i})^{1/2}}\left(1 + \frac{\xi_d^-}{2(\xi_1 + 0\mathrm{i})} + \mathcal{O}(|\xi_1|^{-2})\right). \tag{2.25}$$

Willis and Movchan (1995), Eq. (7.4) define a generalized function $\overline{Q}$ so that

$$[\overline{U}] =: \frac{(2\mathrm{i})^{1/2}}{(\xi_1 + 0\mathrm{i})^{1/2}}\left(1 + \frac{\mathrm{i}\overline{Q}(\omega, \xi_2)}{\xi_1 + 0\mathrm{i}} + \mathcal{O}(|\xi_1|^{-2})\right). \tag{2.26}$$

Thus, in the present context,

$$\overline{Q}(\omega, \xi_2) = -\frac{\mathrm{i}\xi_d^-}{2}. \tag{2.27}$$

2.3. Perturbation of the stress intensity factor and the energy release rate

Willis and Movchan (1995) prove that the perturbation ΔK of the stress intensity factor and crack front position for constant unperturbed stress intensity factor K_0 are related so that

$$\frac{\overline{\Delta K}}{\epsilon K_0} = \left(\overline{Q} + \frac{1}{2}m\right)\bar{\phi}, \tag{2.28}$$

where $m = \sqrt{2\pi}M/K_0$, the expansion of the stress ahead of the unperturbed crack being given by

$$\sigma \sim \left\{ \frac{K_0}{\sqrt{2\pi X}} - P(t, x_2) + M\sqrt{X} \right\} H(X) \quad \text{as } X \to 0_+. \tag{2.29}$$

Woolfries and Willis (1999) use (2.28) to deduce that the Fourier transform with respect to (t, x_2) of the energy release rate satisfies

$$\frac{\overline{\Delta G}}{\epsilon G_0} = (\bar{q} + m)\bar{\phi}, \tag{2.30}$$

where

$$\bar{q} = 2\overline{Q} - \frac{\mathrm{i}V\omega}{\delta^2 d^2} \tag{2.31}$$

in the case of elasticity. Morrissey and Rice (1998) gave the relation (2.30) in the case $m = 0$. Since the near-tip stress and displacement fields for viscoelasticity are the same as those for elasticity using the elastic, high-frequency, moduli (see e.g. Freund, 1990), this result may be used here, with

$$\bar{q}(\omega, \xi_2) = -\frac{V}{2\delta^2 d^2 \tau} + \mathrm{i}\left(\frac{\omega^2}{\delta^4 d^2} + \frac{\mathrm{i}\omega}{\delta^4 d^2 \tau} - \frac{V^2}{4\delta^4 d^4 \tau^2} - \frac{\xi_2^2}{\delta^2} \right)^{1/2}. \tag{2.32}$$

2.4. The transfer function h

Renaming ξ_2 as k and x_2 as z, the crack front position and energy release rate are related so that

$$\epsilon\bar{\phi}(\omega, k) = \bar{h}(\omega, k)\frac{\overline{\Delta G}}{G_0}(\omega, k), \tag{2.33}$$

where

$$\bar{h}(\omega, k) = \frac{1}{(m - (V/(2d^2\delta^2\tau))) + (\mathrm{i}/(\delta^2 d^2))f(\omega, k)}, \tag{2.34}$$

and

$$f(\omega, k) = \left(\omega^2 + \frac{\mathrm{i}\omega}{\tau} - \frac{V^2}{4d^2\tau^2} - k^2\delta^2 d^2 \right)^{1/2}. \tag{2.35}$$

The branches of $f(\omega, k)$ are chosen such that $f(\omega, k)$ has positive imaginary part when ω and k are real. The branch points of $f(\omega, k)$ in the complex ω plane for real k are at

$$\omega = -\frac{\mathrm{i}}{2\tau} \pm \frac{\delta}{2}\left(-\frac{1}{\tau^2} + 4k^2 d^2 \right)^{1/2}. \tag{2.36}$$

If $4k^2d^2 < 1/\tau^2$, the branch cut runs along the negative imaginary ω axis, and is given by

$$\Re\omega = 0, \quad -\frac{1}{2\tau} - \frac{\delta}{2}\sqrt{\frac{1}{\tau^2} - 4k^2d^2} < \Im\omega < -\frac{1}{2\tau} + \frac{\delta}{2}\sqrt{\frac{1}{\tau^2} - 4k^2d^2}. \tag{2.37}$$

On the imaginary ω axis, $f(\omega, k)$ is pure imaginary, with positive imaginary part above the branch cut, and negative imaginary part below it. Just to the left of the branch cut, $f(\omega, k)$ is real and positive, and just to the right it is real and negative. If $4k^2d^2 > 1/\tau^2$, the branch cut is

$$\Im\omega = -\frac{\mathrm{i}}{2\tau}, \quad -\frac{\delta}{2}\sqrt{4k^2d^2 - \frac{1}{\tau^2}} < \Re\omega < \frac{\delta}{2}\sqrt{4k^2d^2 - \frac{1}{\tau^2}}. \tag{2.38}$$

Just above the branch cut, $f(\omega,k)$ is pure imaginary with positive imaginary part, and just below, it is pure imaginary with negative imaginary part.

Candidate singularities of $\bar{h}(\omega,k)$ in the complex ω plane are located where $f^2 = (m - V/2\delta^2 d^2\tau)^2\delta^4 d^4$; thus,

$$2\omega = -\frac{\mathrm{i}}{\tau} \pm \left(-\frac{1}{\tau^2} + 4\delta^2 k^2 d^2 - 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2 \tau}\right)\right)^{1/2}. \tag{2.39}$$

In addition, for ω to be a singularity of $\bar{h}(\omega,k)$, we need $\mathrm{sgn}\,\Im f(\omega,k) = \mathrm{sgn}(m - V/2\delta^2 d^2\tau)$, and also $\Re f(\omega,k) = 0$. The residue of $\bar{h}(\omega,k)$ about a pole $\omega = \omega_0$ is given by

$$\mathrm{Res}\,(\bar{h}, \omega = \omega_0) = \frac{\delta^4 d^2}{\omega_0 + \mathrm{i}/2\tau}\left(m - \frac{V}{2\delta^2 d^2\tau}\right). \tag{2.40}$$

If $4k^2d^2 < 1/\tau^2$, the conditions for a singularity are met only if the singularity lies on the imaginary ω axis away from the branch cut. That the singularity has no real part requires

$$\frac{1}{\tau^2} - 4\delta^2 k^2 d^2 + 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2\tau}\right) > 0 \tag{2.41}$$

and also, for the singularity to lie outside the branch cut, in addition,

$$\frac{V^2}{d^2\tau^2} + 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2\tau}\right) > 0. \tag{2.42}$$

If both these conditions are met, then only one of the two candidate singularities is allowed, namely at $\omega = \omega_0$, where

$$\begin{aligned} \omega_0 &:= -\frac{\mathrm{i}}{2\tau} + \frac{\mathrm{i}}{2}\,\mathrm{sgn}\left(m - \frac{V}{2\delta^2 d^2\tau}\right)\Omega(k), \\ \Omega(k) &= \sqrt{\frac{1}{\tau^2} - 4\delta^2 k^2 d^2 + 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2\tau}\right)}. \end{aligned} \tag{2.43}$$

Note that if $m(m - V/d^2\delta^2\tau) > k^2$, then ω_0 has positive imaginary part. Put another way, if $m > V/d^2\delta^2\tau$, then ω_0 always has positive imaginary part for some range of values of k.

If $4k^2d^2 > 1/\tau^2$, any singularity must lie either on the branch cut of $\bar{f}(\omega,k)$ or on the real ω axis. The latter is true if

$$\frac{1}{\tau^2} - 4\delta^2 k^2 d^2 + 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2\tau}\right) > 0 \tag{2.44}$$

and then the correct root is given by $\omega = \omega_0$, where

$$\omega_0 = -\frac{\mathrm{i}}{2\tau} + \frac{\mathrm{i}}{2}\,\mathrm{sgn}\left(m - \frac{V}{2\delta^2 d^2\tau}\right)\Omega(k). \tag{2.45}$$

Again, if $m > V/\delta^2 d^2\tau$, ω_0 always has positive imaginary part for some range of values of k. On the other hand, if

$$\frac{1}{\tau^2} - 4\delta^2 k^2 d^2 + 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2\tau}\right) < 0, \tag{2.46}$$

and, in addition to this,

$$\frac{V^2}{d^2\tau^2} + 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2 \tau}\right) > 0, \tag{2.47}$$

then two roots on one side of the branch cut are allowed, given by ω_1 which is given the appropriate small imaginary part,

$$\omega_1 = -\frac{\mathrm{i}}{2\tau} \pm \Omega_1(k) + 0\mathrm{i}\,\mathrm{sgn}\left(m - \frac{V}{2\delta^2 d^2 \tau}\right), \tag{2.48}$$

where

$$\Omega_1(k) = \sqrt{-\frac{1}{\tau^2} + 4\delta^2 k^2 d^2 - 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2 \tau}\right)}. \tag{2.49}$$

Evaluation of the transfer function h is described in Appendix B.

A term that grows exponentially with time for some range of values of k is present if and only if

$$m > \frac{V}{\delta^2 d^2 \tau}. \tag{2.50}$$

If this is not the case, then $\bar{h}(\omega, k)$ is analytic in the upper half of the complex ω plane for all k. In inverting the transform, both ω and k may be taken real, and no singularities are encountered on the axes of integration.

2.5. Behaviour near the wave front

Making the substitutions

$$u = \omega - k\delta d, \tag{2.51}$$

$$v = \omega + k\delta d \tag{2.52}$$

in Eq. (2.34), and taking $m = 0$, the following equation is obtained for the transfer function $h(t, z)$,

$$h(t,z) = \frac{\delta}{8\pi^2}\int\int \frac{\exp\left(-\frac{\mathrm{i}}{2}u\left(t - \frac{z}{\delta d}\right)\right)\exp\left(-\frac{\mathrm{i}}{2}v\left(t + \frac{z}{\delta d}\right)\right)}{-A + \mathrm{i}\left(uv + \frac{\mathrm{i}}{2\tau}(u+v) - A^2\right)^{1/2}}\,\mathrm{d}u\,\mathrm{d}v, \tag{2.53}$$

where

$$A := \frac{V}{2d\tau}. \tag{2.54}$$

We are interested in the limiting behaviour of $h(t, z)$ as z tends to a point on the wavefront, for t fixed. Defining, therefore,

$$\zeta := \frac{1}{2}\left(t - \frac{z}{\delta d}\right) \tag{2.55}$$

we obtain

$$h(t, \delta d(t - 2\zeta)) = \lim_{\zeta\to 0}\frac{\delta}{8\pi^2}\int\int \frac{\exp(-\mathrm{i}\zeta u)\exp(-\mathrm{i}vt)}{-A + \mathrm{i}\left(uv + \frac{\mathrm{i}}{2\tau}(u+v) - A^2\right)^{1/2}}\,\mathrm{d}u\,\mathrm{d}v. \tag{2.56}$$

Replacing the variable u by

$$s = \left(u\left(v + \frac{\mathrm{i}}{2\tau} \right) + \frac{\mathrm{i}v}{2\tau} - A^2 \right)^{1/2}, \tag{2.57}$$

and choosing the branch with positive imaginary part means that

$$s \sim \mathrm{i}\left(v + \frac{\mathrm{i}}{2\tau} \right)^{1/2} |u|^{1/2} \quad \text{as } u \to -\infty \tag{2.58}$$

$$s \sim \left(v + \frac{\mathrm{i}}{2\tau} \right)^{1/2} |u|^{1/2} \quad \text{as } u \to +\infty, \tag{2.59}$$

and the contour of integration may be taken such that $s/(v + \mathrm{i}/2\tau)^{1/2}$ lies entirely in the first quadrant of the complex s-plane. Then,

$$h(t, \delta d(t - 2\zeta)) = \lim_{\zeta \to 0} \frac{\delta}{4\pi^2} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{\zeta v}{2\tau(v+\mathrm{i}/2\tau)} \right) \exp\left(-\frac{\mathrm{i}\zeta A^2}{v+\mathrm{i}/2\tau} \right) \exp(-\mathrm{i}vt)}{v + \mathrm{i}/2\tau} I(\zeta')\,\mathrm{d}v, \tag{2.60}$$

where

$$\zeta' := \frac{\zeta}{v + \mathrm{i}/2\tau} \tag{2.61}$$

and

$$I(\zeta') = \int \frac{s \exp(-\mathrm{i}\zeta' s^2)}{-A + \mathrm{i}s}\,\mathrm{d}s, \tag{2.62}$$

where s follows some contour in the upper half-plane. By differentiating with respect to ζ', $I(\zeta')$ satisfies the differential equation

$$\frac{\partial I}{\partial \zeta'} + 2AI = -\mathrm{i}\frac{\partial}{\partial \zeta'} \int \exp(-\mathrm{i}\zeta' s^2)\,\mathrm{d}s. \tag{2.63}$$

Substituting

$$w = \frac{s^2}{v + \mathrm{i}/2\tau}, \tag{2.64}$$

the contour of integration on the right hand side of (2.63) may be deformed to lie just above the real axis, since $w \to \pm\infty + 0\mathrm{i}$ as $u \to \pm\infty$. Hence,

$$\begin{aligned} \int \exp(-\mathrm{i}\zeta' s^2)\,\mathrm{d}s &= \frac{1}{2}\left(v + \frac{\mathrm{i}}{2\tau} \right)^{1/2} \int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}} \exp\left(\frac{-\mathrm{i}\zeta w}{w^{1/2}} \right) \mathrm{d}w \\ &= \sqrt{\frac{\pi}{\zeta}}\left(v + \frac{\mathrm{i}}{2\tau} \right)^{1/2} \exp\left(-\frac{\pi \mathrm{i}}{4} \right) H(\zeta), \end{aligned} \tag{2.65}$$

(see e.g. Gelf'and and Shilov, 1964). Differentiating with respect to ζ',

$$-\mathrm{i}\frac{\partial}{\partial \zeta'} \int_{-\infty}^{\infty} \exp(-\mathrm{i}\zeta' s^2)\,\mathrm{d}s = \frac{\mathrm{i}}{2} \exp\left(-\frac{\pi \mathrm{i}}{4} \right) \frac{\sqrt{\pi}}{(\zeta')^{3/2}} H(\zeta), \tag{2.66}$$

and then solving for $I(\zeta')$,

$$I(\zeta') = \frac{\mathrm{i}\sqrt{\pi}}{2}\exp\left(-\frac{\pi \mathrm{i}}{4}\right)\left\{-\frac{2}{(\zeta')^{1/2}} + 2\sqrt{A}\sqrt{2\pi}\,\mathrm{erfi}\left(\sqrt{2}\sqrt{A}\sqrt{\zeta'}\right)\right\}H(\zeta) + \mathrm{const}\,\exp(-2A\zeta')H(\zeta)$$
$$\sim -\exp\left(-\frac{\pi \mathrm{i}}{4}\right)\frac{\mathrm{i}\sqrt{\pi}}{(\zeta')^{1/2}}H(\zeta) + \mathcal{O}(1) \tag{2.67}$$

as $\zeta' \to 0$. Taking the leading term only, and putting $\zeta = 0$ in the terms in the exponential of (2.60),

$$h(t, \delta d(t - 2\zeta)) \sim -\frac{\mathrm{i}\delta\sqrt{\pi}}{4\pi^2\sqrt{\zeta}}\exp\left(-\frac{\pi \mathrm{i}}{4}\right)H(\zeta)\int_{-\infty}^{\infty}\frac{\exp(-\mathrm{i}vt)}{(v + \mathrm{i}/2\tau)^{1/2}}\,\mathrm{d}v. \tag{2.68}$$

Lowering the v contour to run just above the branch cut of the square root function,

$$h(t, \delta d(t - 2\zeta)) \sim -\frac{\mathrm{i}\delta\sqrt{\pi}}{4\pi^2\sqrt{\zeta}}\exp\left(-\frac{\pi \mathrm{i}}{4}\right)\exp\left(-\frac{t}{2\tau}\right)H(\zeta)\int_{-\infty+0\mathrm{i}}^{\infty+0\mathrm{i}}\frac{\exp(-\mathrm{i}vt)}{(v)^{1/2}}\,\mathrm{d}v, \tag{2.69}$$

and finally,

$$h(t, \delta d(t - 2\zeta)) \sim -\frac{\delta}{2\pi}\frac{\exp\left(-\frac{t}{2\tau}\right)}{\sqrt{\zeta t}}H(t)H(\zeta) \tag{2.70}$$

as $\zeta \to 0$. Note that putting $\tau = \infty$ in this expression regains the limiting behaviour near the wavefront for the transfer function $h(t, z)$ in the elastic case (see Morrissey and Rice, 1998; Woolfries and Willis, 1999). Fig. 1 shows values of $h(t, z)$ obtained by numerical inversion of the Fourier transform $\bar{h}(\omega, k)$ given by

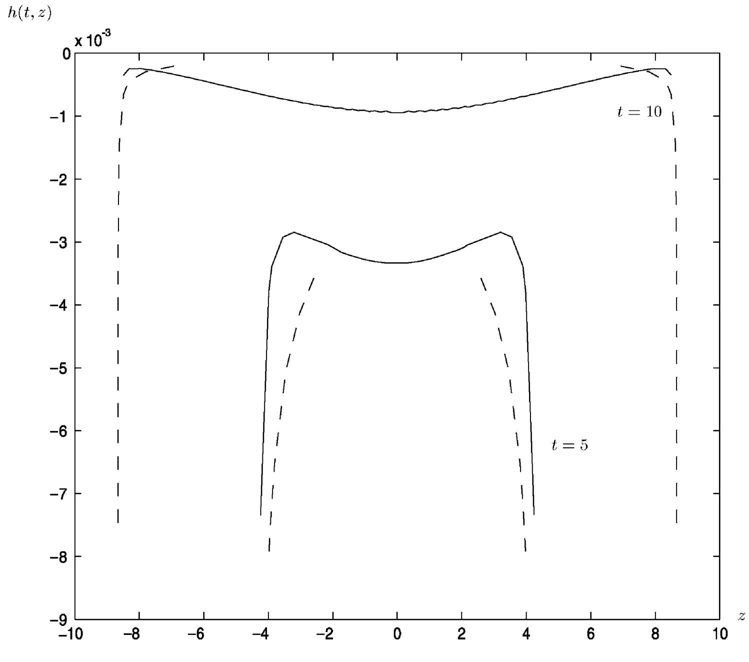

Fig. 1. A plot of the transfer function $h(t, z)$ versus z for the scalar model problem, for $t = 5$ and 10. Units are chosen such that the relaxation time τ and the elastic wave speed d are unity. The crack speed is $V = 0.5$. The dashed lines show the limiting behaviour near the wavefront given by (2.70).

(2.34), for $\tau = 1$, with $m = 0$. The dashed lines show the limiting behaviour (2.70). As $\zeta \to 0$, the solid and dashed lines approach each other; the numerical simulation does not allow us to obtain highly accurate numerical results near the limit points. The crack front position could be calculated from the energy release rate ΔG in a particular problem as in Woolfries and Willis (1999) via (2.33),

$$\epsilon\phi(t,z) = h(t,z) * \frac{\Delta G}{G_0}(t,z), \tag{2.71}$$

where $*$ means convolution over t and z.

3. Mode I crack in a viscoelastic medium

3.1. Governing equations. Wiener–Hopf factorization

The constitutive equation for a viscoelastic solid in three dimensions is taken as

$$\sigma_{ij}(t,\mathbf{x}) = \int L(t-t')\left\{\lambda\delta_{ij}e_{kk}(t',\mathbf{x}) + 2\mu e_{ij}(t',\mathbf{x})\right\}\mathrm{d}t', \tag{3.1}$$

where λ and μ are constants. Fourier transforming with respect to time only,

$$\tilde{\sigma}_{ij}(\omega,\mathbf{x}) = \widetilde{L}(\omega)\left\{\lambda\delta_{ij}\tilde{e}_{kk}(\omega,\mathbf{x}) + 2\mu\tilde{e}_{ij}(\omega,\mathbf{x})\right\}. \tag{3.2}$$

In this section, we specialize to the Maxwell fluid (2.18), so that $\widetilde{L}$ is given by

$$\widetilde{L}(\omega) = \frac{\omega\tau}{\omega\tau + \mathrm{i}}. \tag{3.3}$$

The equations for the behaviour of stress and displacement fields, Fourier transformed with respect to time in the fixed frame of reference, may be obtained by replacing λ by $\widetilde{L}(\omega)\lambda$ and μ by $\widetilde{L}(\omega)\mu$. In particular, Appendix A of Willis and Movchan (1995) follows through to give the equation for the displacement jump $[U_3]$ across $x_3 = 0$ in terms of the traction Σ_{33} on $x_3 = 0$ for Mode I,

$$\widetilde{\Sigma}_{33}(\omega',\xi_1,\xi_2) = \frac{\mathrm{i}\mu b^2\left(\widetilde{L}(\omega')\right)^2 D(\omega',\xi_1,\xi_2)}{2\omega'^2\left(\dfrac{\omega'^2}{a^2\widetilde{L}(\omega')} - |\xi|^2\right)^{1/2}}\left[\widetilde{U}_3\right](\omega',\xi_1,\xi_2), \tag{3.4}$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2$, and the constants a, b denote the relevant elastic dilatational and shear wave speeds,

$$a^2 = (\lambda + 2\mu)/\rho, \quad b^2 = \mu/\rho. \tag{3.5}$$

Following Willis and Movchan (1995), ω' is to be set as $\omega' = \omega - V\xi_1$. The function $D(\omega',\xi_1,\xi_2)$ is given by

$$D(\omega',\xi_1,\xi_2) = 4|\xi|^2\left(\frac{\omega'^2}{a^2\widetilde{L}(\omega')} - |\xi|^2\right)^{1/2}\left(\frac{\omega'^2}{b^2\widetilde{L}(\omega')} - |\xi|^2\right)^{1/2} + \left(\frac{\omega'^2}{b^2\widetilde{L}(\omega')} - 2|\xi|^2\right)^2. \tag{3.6}$$

The factor D has a double zero at $\omega' = 0$. The equation $D = 0$ is also satisfied when

$$\frac{\omega'^2}{|\xi|^2} = c^2\widetilde{L}(\omega'), \tag{3.7}$$

where the constant c satisfies the Rayleigh equation based on a and b. From (2.20), Eq. (3.7) is satisfied when $\xi_1 = \xi_c^\pm$, where $\xi_c^\pm$ is defined as in (2.20), with d replaced by c. The values $\xi_a^\pm$ and $\xi_b^\pm$ are similarly defined. From the work in the previous section, there is one root in either half of the complex ξ_1 plane, away from the real axis. Note that the presence of the term $\mathrm{i}\omega/\tau$ ensures that there is no need for the addition to ω of a small imaginary part to separate the roots and ensure causality, as was necessary for the case of elasticity in Willis and Movchan (1995).

Eq. (2.20) also provides the factorization

$$\left(\frac{\omega'^2}{a^2\widetilde{L}(\omega')} - |\xi|^2\right)^{1/2} = \mathrm{i}\alpha\left(\xi_1 - \xi_a^+\right)^{1/2}\left(\xi_1 - \xi_a^-\right)^{1/2}, \tag{3.8}$$

where the constant $\alpha = (1 - V^2/a^2)^{1/2}$. Hence, $D(\omega', \xi_1, \xi_2)$ has branch points at $\xi_1 = \xi_a^{\pm}$ and $\xi_1 = \xi_b^{\pm}$. Also,

$$D\left(\omega', \xi_1, \xi_2\right) \sim -\xi_1^4 R(V) \tag{3.9}$$

as $\xi_1 \to \infty$, where

$$R(V) = 4\alpha\beta - (1 + \beta^2)^2, \tag{3.10}$$

and the constant $\beta = (1 - V^2/b^2)^{1/2}$. Therefore, defining

$$T(\omega, \xi_1, \xi_2) := -\frac{D\left(\omega', \xi_1, \xi_2\right)V^2}{\left(\xi_1 - \xi_c^+\right)\left(\xi_1 - \xi_c^-\right)\omega'^2 R(V)} \tag{3.11}$$

ensures that T is analytic and non-zero in a strip containing the real ξ_1 axis and tends to 1 as $\xi_1 \to \infty$. Hence,

$$T(\omega, \xi_1, \xi_2) = T_+(\omega, \xi_1, \xi_2)T_-(\omega, \xi_1, \xi_2), \tag{3.12}$$

where

$$T_{\pm}(\omega, \xi_1, \xi_2) = \exp\left\{\frac{-1}{2\pi\mathrm{i}}\int_{C_{\pm}} \frac{\ln T\left(\omega, \xi_1', \xi_2\right)}{\xi_1' - \xi_1}\,\mathrm{d}\xi_1'\right\}. \tag{3.13}$$

The function $\ln T(\omega, \xi_1', \xi_2)$ is analytic in the complex ξ_1' plane, with branch cuts $B_{\pm}$ joining $\xi_a^{\pm}$ and $\xi_b^{\pm}$ in the upper or lower half of the complex ξ_1' plane. The contours $C_{\pm}$ enclose these branch cuts. The Wiener–Hopf factorization of (3.4) is therefore

$$\frac{2\beta\overline{\Sigma}_{33}\left(\xi_1 - \xi_a^+\right)^{1/2}}{\mu b^2\left(\widetilde{L}(\omega')\right)^2\left(\xi_1 - \xi_c^+\right)T_-} = -\frac{\left(\xi_1 - \xi_c^-\right)R(V)T_+}{\left(\xi_1 - \xi_a^-\right)^{1/2}V^2}\left[\overline{U}_3\right]. \tag{3.14}$$

The right hand side is analytic in an upper half-plane, the left hand side in a lower half-plane. By the conditions imposed on $[U_3]$, outlined in the previous section, $[\overline{U}_3] \sim (2\mathrm{i})^{1/2}(\xi_1 + 0\mathrm{i})^{-1/2}$ as $\xi_1 \to \infty$. Hence, the right hand side is a bounded entire function, and therefore must be a constant, by Liouville's theorem. Thus,

$$\left[\overline{U}_3\right] = \frac{(2\mathrm{i})^{1/2}\left(\xi_1 - \xi_a^-\right)^{1/2}}{\left(\xi_1 - \xi_c^-\right)T_+} \tag{3.15}$$

(c.f. Willis and Movchan, 1995, Eq. (4.16)).

3.2. Analysis of the transfer function

The generalized function $\overline{Q}(\omega, \xi_2)$ relating crack front position to stress intensity factor is defined by

$$[\overline{U}_3] \sim \frac{(2\mathrm{i})^{1/2}}{(\xi_1 + 0\mathrm{i})^{1/2}} \left\{ 1 + \frac{\mathrm{i}\overline{Q}}{(\xi_1 + 0\mathrm{i})} \right\} \tag{3.16}$$

as $\xi_1 \to \infty$ (see Willis and Movchan, 1995, Eq. (7.4)). Therefore,

$$\mathrm{i}\overline{Q} = -\frac{1}{2}\xi_a^- + \xi_c^- - \frac{1}{2\pi\mathrm{i}} \int_{C_-} \ln T\left(\omega, \xi_1', \xi_2\right) \mathrm{d}\xi_1' \tag{3.17}$$

(c.f. Willis and Movchan, 1995, Eq. (7.6)). The integral term in the above expression is treated in Appendix C. The function $\overline{Q}(\omega, k)$ participates in the relation

$$\frac{\overline{\Delta K}}{\epsilon K_0} = \left(\overline{Q} + \frac{1}{2}m\right)\bar{\phi} \tag{3.18}$$

and is therefore designated a transfer function. The relationship between stress intensity factor and energy release rate for Mode I viscoelasticity is given by (e.g. Freund, 1990)

$$G = f(v)K^2/2\mu, \tag{3.19}$$

where $f(v)$ is given by

$$f(v) = \frac{\alpha v^2}{(1-\nu)R(v)b^2}. \tag{3.20}$$

Here, ν is Poisson's ratio, $\nu = \lambda/2(\lambda + \mu)$. The relationship between the crack front position ϕ and first-order perturbation in energy release rate ΔG is obtained from (3.19) by substituting $v = V + \epsilon\dot{\phi}$, $G = G_0 + \Delta G$, $K = K_0 + \Delta K$, taking order ϵ perturbations and using (3.18). The result is

$$\frac{\overline{\Delta G}}{\epsilon G_0}(\omega, k) = \left(\bar{q}(\omega, k) + m\right)\bar{\phi}(\omega, k), \tag{3.21}$$

where

$$\bar{q}(\omega, k) = 2\overline{Q}(\omega, k) - \mathrm{i}\omega\frac{f'(V)}{f(V)}. \tag{3.22}$$

Correspondingly,

$$\epsilon\bar{\phi}(\omega, k) = \frac{\overline{\Delta G}}{\epsilon G_0}(\omega, k)\frac{1}{\left(\bar{q}(\omega, k) + m\right)}. \tag{3.23}$$

Ramanathan (1997) (also reported in Ramanathan and Fisher, 1997) has numerically evaluated the transfer function $\bar{q}(\omega, k)$ from formulae corresponding to those of Willis and Movchan (1995) for the purely elastic case (corresponding to the limit $\tau \to \infty$). In this limit, the transfer function is a homogeneous function of degree 1, and has a zero at a certain real value of ω/k. This gives rise to a singularity in the relationship (3.23) when $m = 0$. Morrissey and Rice (1998) showed that, for the case of mode I elasticity, this corresponds to a persistent propagating mode (crack front wave) for crack growth at constant fracture energy, and found evidence of such a mode in their numerical calculations.

The existence of a singularity on the real axis in (3.23) also affects the crack stability. Ramanathan (1997) describes this as being ‘on the boundary of a regime of stability’, and points out that any dependence of the energy release rate on velocity of crack propagation would resolve this.

We have now a mechanism for resolving this in the case of viscoelasticity. Ramanathan (1997), Fig. 3.2 plots $\bar{q}(\omega,k)/|k|$ against ω^2/k^2 for elasticity. From the fact that this takes both positive and negative real values, and from the fact that in this case $\bar{q}$ is homogeneous of degree 1, we can deduce that, in the case of elasticity, $\bar{q}$ maps the infinite real domain $\{(\omega,k) \in (-\infty,\infty) \times (-\infty,\infty)\}$ onto a region of the complex plane which contains the whole real axis. This gives rise to a singularity of the transfer function given in (3.23) for any m and gives the value of the ratio ω/k at which the zero occurs, corresponding to (non-dispersive) crack front waves in the case $m = 0$. Fig. 2 shows the domain $\mathscr{D}$ of the complex plane defined as $\{\bar{q}(\omega,k) : (\omega,k) \in [0,1] \times [0,1]\}$ for different relaxation times τ. Parameters are chosen such that $c = 1$, $a = 1.8569757$, $b = 1.0721216$ and $V = 0.5$. It should be noted that $\bar{q}$ is symmetric in k and Hermitian, so that $\bar{q}(-\omega,-k) = \bar{q}(-\omega,k) = \bar{q}^*(\omega,k)$. In Fig. 2a, b, c, it is explicit that $\mathscr{D}$ has no common points with the positive real axis. This feature is also present in Fig. 2d, despite the very large relaxation time, and this in fact remains the case for any positive τ. In Fig. 2b, for example, $\mathscr{D}$ does not contain any real number greater

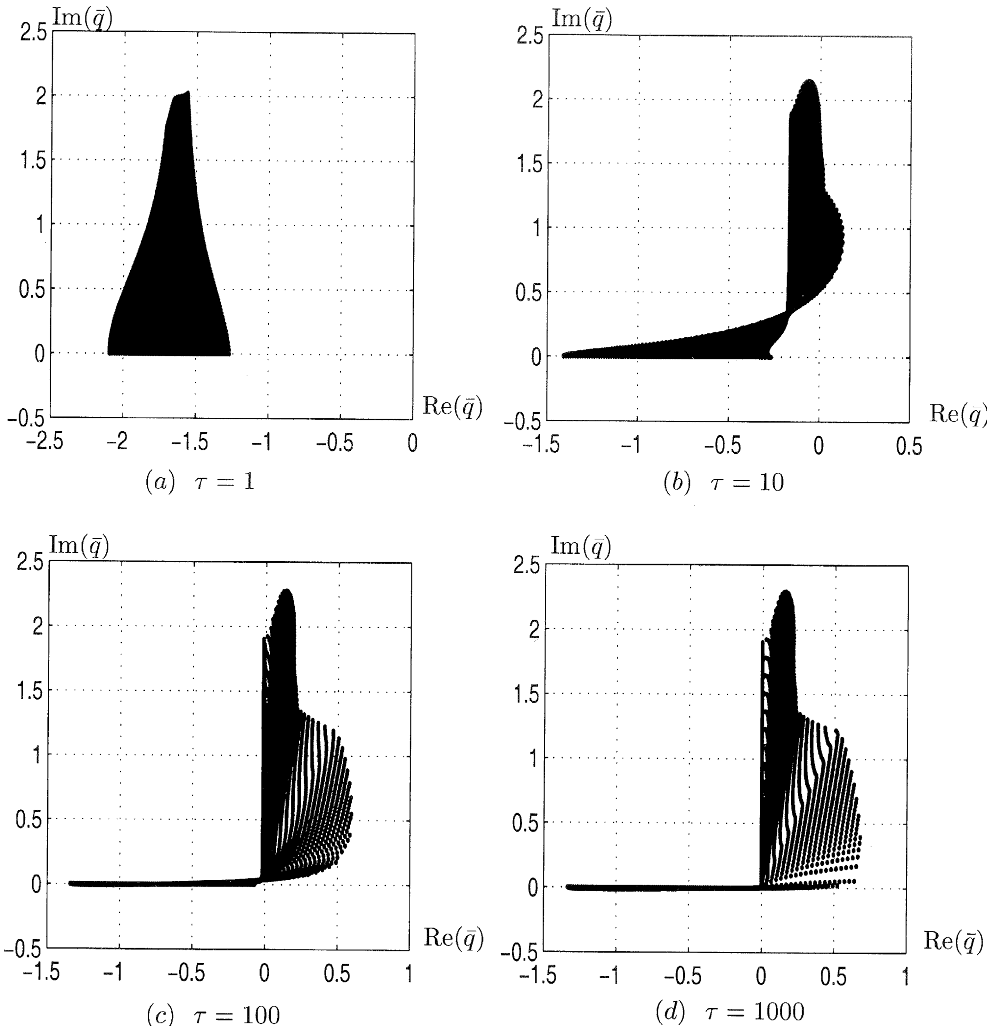

Fig. 2. The region $\mathscr{D} = \{\bar{q}(\omega,k) : (\omega,k) \in [0,1] \times [0,1]\}$. Parameters are chosen such that $c = 1$, $a = 1.856975729$, $b = 1.072121621$ and $V = 0.5$.

than about -0.25. This means that there is no singularity in the relationship (3.23) for $m < 0.25$. In particular, for $m = 0$, the introduction of any amount of viscoelasticity destroys the existence of crack front waves and instability.

Acknowledgement

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Appendix A. Definition of the weight function

To justify the definition in Section 2 of the function $U(t, x_1, x_2, x_3)$, note first that, if functions $f(t, x_1, x_2)$ and $g(t, x_1, x_2)$ are given, and the definitions

$$F(t,X,x_2) := f(t, Vt + X, x_2), \quad G(t,X,x_2) := g(t, Vt + X, x_2) \tag{A.1}$$

are made, then

$$\begin{aligned}(F * G)(t,X,x_2) &= \int \mathrm{d}t' \int \mathrm{d}X' \int \mathrm{d}x_2' f\big(t', Vt' + X', x_2'\big) g\big(t - t', V(t - t') + X - X', x_2 - x_2'\big) \\ &= \int \mathrm{d}t' \int \mathrm{d}x_1' \int \mathrm{d}x_2' f\big(t', x_1', x_2'\big) g\big(t - t', Vt + X - x_1', x_2 - x_2'\big) \\ &= (f * g)(t, Vt + X, x_2) \equiv (f * g)(t, x_1, x_2); \quad x_1 = Vt + X,\end{aligned} \tag{A.2}$$

where the symbol $*$ represents convolution over the natural arguments of the respective functions.

Now Willis (1997), Eq. (4.11) derived, in the context of elastodynamics, the identity

$$([U_i] * \sigma_{i3} - \Sigma_{k3} * [u_k])(t, x_1, x_2) = 0 \tag{A.3}$$

relative to the stationary frame of reference. However, in this frame, it also applies immediately if the response of the medium is linearly viscoelastic: Fourier transforming with respect to t, for example, reproduces the equations of elastodynamics, except that the "elastic constants" now become functions of ω. The result (A.2) shows that the identity (A.3) can equally well be interpreted relative to the moving frame. With slight distortion of the notation,

$$([U_i] * \sigma_{i3} - \Sigma_{k3} * [u_k])(t, X, x_2) = 0. \tag{A.4}$$

The problem for Mode I loading uncouples from those for Modes II and III. Hence, for Mode I, dropping the suffixes,

$$([U] * \sigma - \Sigma * [u])(t, X, x_2) = 0. \tag{A.5}$$

This equation applies equally well to the model problem of Section 2.

Now decompose σ into two parts:

$$\sigma = \sigma_+ + \sigma_-, \tag{A.6}$$

where $\sigma_+ = 0$ for all $X < 0$, and $\sigma_- = \sigma - \sigma_+$. The function $[u]$ is a "$-$" function, in this notation. It is an unknown of the problem. The function σ_- is known, from the boundary conditions. The function σ_+ is to be found. In particular, as $X \to 0_+$,

$$\sigma_+ \sim K(t, x_2)/\sqrt{2\pi X}, \tag{A.7}$$

the stress intensity factor K being so far unknown. However, the function $[U]$ is a "+" function, while Σ is a "−" function. Therefore, when $X > 0$, $\Sigma * [u] = 0$ (since $[u]$ is also a "−" function), and the identity (A.5) reduces to

$$([U] * \sigma_+)(t, X, x_2) = -([U] * \sigma_-)(t, X, x_2); \quad X > 0. \tag{A.8}$$

Finally, the convolution on the left side can be evaluated explicitly when $X \to 0_+$, from just the asymptotic forms (2.13) and (A.7), to give

$$K(t, x_2) = -([U] * \sigma_-)(t, 0, x_2). \tag{A.9}$$

The function $[U](t, X, x_2)$ is thus a "weight function".

Appendix B. Evaluation of the transfer function for a scalar problem

The transfer function $h(t, z)$ is given by

$$h(t,z) = \frac{1}{4\pi^2}\int \exp(-\mathrm{i}kz)\,\mathrm{d}k \int \exp(-\mathrm{i}\omega t)\bar{h}(\omega, k)\,\mathrm{d}\omega, \tag{B.1}$$

where the line of integration in the complex ω plane must pass above all singularities, to ensure causality. Enclosing the contour in the lower half-plane for $t > 0$, we pick up all of the pole contributions away from the branch cuts. The contour still encloses the singularities situated just above or below the branch cut. These can be expressed in terms of a Cauchy principal value plus contributions from the singularity, using the Plemelj formulae.

The branch cut contribution for $4k^2d^2 < 1/\tau^2$ is given by

$$\oint \exp(-\mathrm{i}\omega t)\bar{h}(\omega,k)\,\mathrm{d}\omega = -\frac{2\mathrm{i}}{\delta^2 d^2}\exp(-t/2\tau)\exp\left(-\frac{\delta}{2}\sqrt{\frac{1}{\tau^2} - 4k^2d^2 t}\right) \times \int_0^{\delta\sqrt{\frac{1}{\tau^2}-4k^2d^2}} \frac{\sqrt{s}\sqrt{s + \delta\sqrt{\frac{1}{\tau^2} - 4k^2d^2}}\exp(st)}{\left(m - \frac{V}{2d^2\delta^2\tau}\right)^2 + \frac{s}{\delta^4 d^2}\left(s + \delta\left(\frac{1}{\tau^2} - 4k^2d^2\right)\right)}\,\mathrm{d}s. \tag{B.2}$$

The principal part of the branch cut contribution for $4k^2d^2 > 1/\tau^2$ is given by

$$\oint \exp(-\mathrm{i}\omega t)\bar{h}(\omega,k)\,\mathrm{d}\omega = \frac{2}{\delta^2 d^2}\exp(-t/2\tau)\int_0^{\delta\sqrt{4k^2d^2-\frac{1}{\tau^2}}} \frac{\exp(-\mathrm{i}st)\sqrt{s}\sqrt{\delta\sqrt{4k^2d^2 - \frac{1}{\tau^2}} - s}}{\left(m - \frac{V}{2d^2\delta^2\tau}\right)^2 - \frac{s}{\delta^4 d^2}\left(\delta\sqrt{4k^2d^2 - \frac{1}{\tau^2}} - s\right)}\,\mathrm{d}s. \tag{B.3}$$

Adding all of the contributions, the transfer function $h(t, z)$ is given by

$$h(t,z) = \frac{1}{4\pi^2}\int \exp(-\mathrm{i}kz)\hat{h}(t,k)\,\mathrm{d}k, \tag{B.4}$$

where

$$\hat{h}(t,k) = 4\pi\delta^4 d^2\left|m - \frac{V}{2\delta^2 d^2\tau}\right|\exp(t/2\tau)\exp\left(\frac{1}{2}\mathrm{sgn}\left(m - \frac{V}{2\delta^2 d^2\tau}\right)\Omega(k)t\right)H\left(\frac{1}{\tau^2} - 4k^2\delta^2 d^2\right.$$
$$\left. + 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2\tau}\right)\right)\left[H\left(4k^2d^2 - \frac{1}{\tau^2}\right) + H\left(\frac{1}{\tau^2} - 4k^2d^2\right)H\left(\frac{V^2}{d^2\tau^2} + 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2\tau}\right)\right)\right]$$
$$+ \frac{2\pi\delta^4 d^2\left|m - \frac{V}{2\delta^2 d^2\tau}\right|}{\Omega_1(k)}(\exp(-\mathrm{i}\Omega_1(k)t) - \exp(\mathrm{i}\Omega_1(k)t))H\left(4k^2\delta^2 d^2 - \frac{1}{\tau^2}\right)$$
$$\times H\left(\frac{V^2}{d^2\tau^2} + 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2\tau}\right)\right)H\left(4k^2\delta^2 d^2 - \frac{1}{\tau^2} - 4\delta^2 d^2 m\left(m - \frac{V}{\delta^2 d^2\tau}\right)\right)$$
$$- \frac{2\mathrm{i}}{\delta^2 d^2}H\left(\frac{1}{\tau^2} - 4k^2d^2\right)\exp(-t/2\tau)\exp\left(-\frac{\delta}{2}\sqrt{\frac{1}{\tau^2} - 4k^2d^2t}\right)$$
$$\times\int_0^{\delta\sqrt{\frac{1}{\tau^2}-4k^2d^2}}\frac{\sqrt{s}\sqrt{s + \delta\sqrt{\frac{1}{\tau^2} - 4k^2d^2}}\exp(st)}{\left(m - \frac{V}{2d^2\delta^2\tau}\right)^2 + \frac{s}{\delta^4 d^2}\left(s + \delta\left(\frac{1}{\tau^2} - 4k^2d^2\right)\right)}\mathrm{d}s + \frac{2}{\delta^2 d^2}\exp(-t/2\tau)H\left(4k^2d^2 - \frac{1}{\tau^2}\right)$$
$$\times\int_0^{\delta\sqrt{4k^2d^2-\frac{1}{\tau^2}}}\frac{\exp(-\mathrm{i}st)\sqrt{s}\sqrt{\delta\sqrt{4k^2d^2 - \frac{1}{\tau^2}} - s}}{\left(m - \frac{V}{2d^2\delta^2\tau}\right)^2 - \frac{s}{\delta^4 d^2}\left(\delta\sqrt{4k^2d^2 - \frac{1}{\tau^2}} - s\right)}\mathrm{d}s. \tag{B.5}$$

The last expression in the above equation is interpreted as a Cauchy principal value, if that is required.

Appendix C. Complex integral in the transfer function

The expression (3.17) for the transfer function $\overline{Q}(\omega,k)$ involves the term

$$I(\omega,\xi_2) = \int_{C_-}\ln T\left(\omega,\xi_1',\xi_2\right)\mathrm{d}\xi_1', \tag{C.1}$$

where C_- encloses the branch cut joining ξ_a^- and ξ_b^- in the clockwise direction. The function $T(\omega,\xi_1',\xi_2)$ is defined via (3.11),

$$T\left(\omega,\xi_1',\xi_2\right) := -\frac{D\left(\omega - V\xi_1',\xi_1',\xi_2\right)V^2}{(\xi_1' - \xi_c^+)(\xi_1' - \xi_c^-)(\omega - V\xi_1')^2 R(V)}, \tag{C.2}$$

with D defined by (3.6). The denominator takes the same values on both sides of the branch cut. Above the branch cut, parameterized by

$$\xi_1' = \xi_a^- + (s - 0\mathrm{i})(\xi_b^- - \xi_a^-), \tag{C.3}$$

the numerator takes the value $Y - X$, where

$$X = 4\alpha\beta\mathrm{i}\left(\xi_1' - \xi_a^+\right)^{1/2}\left(\xi_1' - \xi_b^+\right)^{1/2}\left(\xi_1'^2 + \xi_2^2\right)(\xi_a^- - \xi_b^-)\sqrt{s(1-s)},$$
$$Y = \left[\xi_1'^2 + \xi_2^2 + \beta^2(1-s)\left(\xi_1' - \xi_b^+\right)(\xi_a^- - \xi_b^-)\right]^2.$$

Below the branch cut, the numerator takes the value $Y + X$. In the expressions for Y and X, $\xi_1'(s)$ unambiguously takes the values

$$\xi_1'(s) = \xi_a^- + s(\xi_b^- - \xi_a^-). \tag{C.4}$$

The integral I is given by

$$I(\omega, \xi_2) = \left(\xi_b^- - \xi_a^-\right) \int_0^1 \ln\left(\frac{Y - X}{Y + X}\right) \mathrm{d}s, \tag{C.5}$$

where $\xi_1'(s)$ takes the values (C.4) and $\xi_a^\pm$ and $\xi_b^\pm$ are functions of ω and ξ_2 given by (2.21) with the appropriate replacements.

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